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Existence of Positive Radial Solutions for Elliptic Systems

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1. INTRODUCTION

In this paper, we consider the existence of positive radial solutions of the elliptic system

$$\begin{aligned} &\Delta u + a(|x|) f(u,v) = 0, \\ &\Delta v + b(|x|) g(u,v) = 0, \qquad R_1 < |x| < R_2,\ x \in \mathbb{R}^N,\ N \geq 2 \end{aligned} \tag{1.1}$$

with one of the following sets of boundary conditions

$$u = v = 0 \qquad \text{on } |x| = R_1,\ |x| = R_2, \tag*{$(1.2)_a$}$$

$$u = v = 0 \qquad \text{on } |x| = R_1,\ \frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} = 0 \text{ on } |x| = R_2, \tag*{$(1.2)_b$}$$

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial r} = 0 \qquad \text{on } |x| = R_1,\ u = v = 0 \text{ on } |x| = R_2, \tag*{$(1.2)_c$}$$

where $f\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $g\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, $a\colon \mathbb{R}^+ \to \mathbb{R}^+$, $b\colon \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions, and $\mathbb{R}^+ = [0, \infty)$.

The existence and uniqueness of positive radial solutions for scalar equation

$$\Delta u + h(|x|)\bar{f}(u) = 0, \qquad R_1 \leq |x| \leq R_2, \qquad x \in \mathbb{R}^N,\ N \geq 2, \tag{1.3}$$

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have been widely studied (see [1, 5–7]). In [7], Wang studied (1.3) with one of the following sets of boundary conditions

$$u = 0 \qquad \text{on } |x| = R_1,\ |x| = R_2, \tag{1.4$_\mathrm{a}$}$$

$$u = 0 \qquad \text{on } |x| = R_1,\ \frac{\partial u}{\partial r} = 0 \text{ on } |x| = R_2, \tag{1.4$_\mathrm{b}$}$$

$$\frac{\partial u}{\partial r} = 0 \qquad \text{on } |x| = R_1,\ u = 0 \text{ on } |x| = R_2. \tag{1.4$_\mathrm{c}$}$$

He proved the following results.

Assume $\bar{f}: \mathbb{R}^+ \to \mathbb{R}^+$, $h: \mathbb{R}^+ \to \mathbb{R}^+$ are continuous, and h is not identically zero in any finite subinterval of $(0, \infty)$. If $\bar{f}$ satisfies

$$\lim_{u \to 0} \frac{f(u)}{u} = \infty, \qquad \lim_{u \to \infty} \frac{f(u)}{u} = 0, \tag{1.5}$$

then (1.3)–(1.4) has a positive radial solution for any annulus $R_1 < |x| < R_2$.

The purpose of this paper is to give an extension of above-mentioned result of system (1.1)–(1.2). The proof of our main result (see Section 2) makes use of the following fixed point theorem.

FIXED POINT THEOREM [4]. *Let X be a Banach space, let K be a cone in X, let $0 < r < R$ be real numbers, let $D = \{x \in K : r \le \|x\| \le R\}$, and let $T: D \to K$ be a compact continuous operator such that*

(i) $x \in D$, $\|x\| = R$, $Tx = \lambda x \Rightarrow \lambda \le 1$;

(ii) $x \in D$, $\|x\| = r$, $Tx = \lambda x \Rightarrow \lambda \ge 1$;

(iii) $\inf_{\|x\| = r} \|Tx\| > 0$.

Then T has a fixed point in D.

2. MAIN RESULT

For $(x, y) \in \mathbb{R}^2$, denote $|(x, y)| = \max\{|x|, |y|\}$. The following conditions will be assumed in this section:

(H_1) $f: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $g: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, $a: \mathbb{R}^+ \to \mathbb{R}^+$, $b: \mathbb{R}^+ \to \mathbb{R}^+$ are continuous.

(H_2) a and b are not identically zero in any finite subinterval of $(0, \infty)$.

(H_3)

$$\lim_{|(x, y)| \to 0} \frac{f(x, y)}{|(x, y)|} = \infty \tag{2.1}$$

and

$$\lim_{|(x,y)|\to 0} \frac{g(x,y)}{|(x,y)|} = \infty. \tag{2.2}$$

(H_4) Either

(i) f and g are bounded functions, or

(ii) g is bounded and f is unbounded and

$$\lim_{|(x,y)|\to \infty} \frac{f(x,y)}{|(x,y)|} = 0, \tag{2.3}$$

or

(iii) g is increasing in each variable for any value of the other variable, f is unbounded, and

$$\lim_{|(x,y)|\to \infty} \frac{f(x,y)}{|(x,y)|} = 0 = \lim_{|(x,y)|\to \infty} \frac{g(x,y)}{|(x,y)|}, \tag{2.4}$$

or

(iv) f is bounded, g is unbounded, and

$$\lim_{|(x,y)|\to \infty} \frac{g(x,y)}{|(x,y)|} = 0,$$

or

(v) f is increasing in each variable for any value of the other variable, g is unbounded, and (2.4) holds.

THEOREM 1. *Assume* (H_1)–(H_4) *hold. Then* (1.1)–(1.2) *has a positive radial solution for any annulus* $R_1 < |x| < R_2$.

Remark 2. Theorem 1 is an extension of Theorem 1 of [7], which deals with the scalar equation (1.3) with the boundary conditions (1.4).

3. PRELIMINARIES

In view of the spherical symmetry of $g(|x|)$, we seek a positive radial solution $u = u(r), v = v(r)$ to (1.1). Therefore we write (1.1)–(1.2) in the form

$$u''(r) + \frac{N-1}{r}u'(r) + a(r)f(u(r), v(r)) = 0,$$

$$v''(r) + \frac{N-1}{r}v'(r) + b(r)g(u(r), v(r)) = 0, \qquad R_1 < x < R_2, \quad (3.1)$$

$$u(R_1) = v(R_1) = u(R_2) = v(R_2) = 0, \qquad (3.2)_a$$

$$u(R_1) = v(R_1) = u'(R_2) = v'(R_2) = 0, \qquad (3.2)_b$$

$$u'(R_1) = v'(R_1) = u(R_2) = v(R_2) = 0, \qquad (3.2)_c$$

Let $s = -\int_r^{R_2}(1/t^{N-1})\,dt$ and $w(s) = u(r(s))$ and $z(s) = v(r(s))$, then (3.1)–(3.2) can be rewritten as

$$w''(s) + r^{2(N-1)}(s)a(r(s))f(w(s), z(s)) = 0,$$

$$z''(s) + r^{2(N-1)}(s)b(r(s))g(w(s), z(s)) = 0, \qquad m < s < 0,$$

$$w(m) = z(m) = w(0) = z(0) = 0,$$

$$w(m) = z(m) = w'(0) = z'(0) = 0,$$

$$w'(m) = z'(m) = w(0) = z(0) = 0,$$

where $m = -\int_{R_1}^{R_2}(1/t^{N-1})\,dt$.

To be obvious, again let $t = (m - s)/m$ and $\varphi(t) = \omega(s)$ and $\psi(t) = z(s)$. Then (3.1)–(3.2) can also be written as

$$\varphi''(t) + A(t)f(\varphi(t), \psi(t)) = 0,$$

$$\psi''(t) + B(t)g(\varphi(t), \psi(t)) = 0, \qquad (3.3)$$

$$\varphi(0) = \psi(0) = \varphi(1) = \psi(1) = 0, \qquad (3.4)_a$$

$$\varphi(0) = \psi(0) = \varphi'(1) = \psi'(1) = 0, \qquad (3.4)_b$$

$$\varphi'(0) = \psi'(0) = \varphi(1) = \psi(1) = 0, \qquad (3.4)_c$$

where $A(t) = m^2 r^{2(N-1)}(m(1-t))a(r(m(1-t)))$, $B(t) = m^2 r^{2(N-1)}(m(1-t))b(r(m(1-t)))$. It is easy to check that $A(t)$ and $B(t)$ also satisfy (H_2).

From now on, we concentrate on (3.3)–(3.4). Indeed, if we prove that there exists a positive solution to (3.3)–(3.4) for any $m \neq 0$, then (1.1)–(1.2) has a positive radial solution for any annulus.

In the proof of Theorem 1, we will use the following two lemmas.

LEMMA 3. *If $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a unbounded continuous function, then the set*

$$\Sigma_f = \left\{ l \in \mathbb{R}^+ \,\middle|\, \begin{array}{ll} \max\limits_{[0,l;0,l]} f(x,y) = f(l,\bar{y}) & \text{for some } \bar{y} \in [0,l], \text{ or} \\ \max\limits_{[0,l;0,l]} f(x,y) = f(\bar{x},l) & \text{for some } \bar{x} \in [0,l] \end{array} \right\} \tag{3.5}$$

is unbounded.

LEMMA 4. *Let $h \in L^1(0,1)$ and u be a solution of the equation $-u''(x) = h(x)$ with either $u(0) = u(1) = 0$ or $u(0) = u'(1) = 0$ or $u'(0) = u(1) = 0$. If $u \not\equiv 0$, then $u(x) > 0$ for all $x \in (0,1)$.*

The proofs of Lemmas 3 and 4 follow from elementary arguments and are omitted.

4. PROOF OF THEOREM 1

In what follows X will denote the space of continuous functions $u : [0,1] \to \mathbb{R}$ with the norm $\|u\| = \max_{0 \le x \le 1} |u(x)|$.

First consider (3.3)–$(3.4)_a$. Let

$$K_1 = \{ \varphi \in X : \varphi \text{ has graph concave down, } \varphi \ge 0, \ \varphi(0) = \varphi(1) = 0 \};$$

then for $\varphi \in K_1$, we have that $\varphi(t) \ge \frac{1}{4} \|\varphi\|$ for $t \in [\frac{1}{4}, \frac{3}{4}]$.

In $X \times X$ consider the maximum norm and define

$$T : K_1 \times K_1 \to K_1 \times K_1$$

given by

$$T(\varphi, \psi)(t) = \left(\int_0^1 G(t,s) A(s) f(\varphi(s), \psi(s)) \, ds, \right.$$
$$\left. \int_0^1 G(t,s) B(s) g(\varphi(s), \psi(s) \, ds \right),$$

where

$$G(t,s) = \begin{cases} t(1-s), & t \le s, \\ s(1-t), & t > s. \end{cases}$$

It is easy to see $G(t,s) \le G(s,s)$.

Observe that, if we write

$$T(\varphi, \psi) = (\varphi_1, \psi_1),$$

then φ_1, ψ_1 are continuously differentiable on $[0,1]$, twice differentiable on $(0,1)$, and

$$\varphi_1'(t) = -\int_0^t sA(s) f(\varphi(s), \psi(s))\, ds + \int_t^1 (1-s) A(s) f(\varphi(s), \psi(s))\, ds,$$
$$\varphi_1''(t) = -A(s) f(\varphi(t), \psi(t)) \le 0;$$

likewise for ψ_1. This implies that T is well defined and the Arzela–Ascoli theorem implies that T is a compact mapping. It is easy to check that (3.3)–$(3.4)_{\mathrm{a}}$ is equivalent to the fixed point problem in $K_1 \times K_1$,

$$(\varphi, \psi) = T(\varphi, \psi).$$

From (H_3), there is a $r > 0$ such that

$$f(x,y) \ge M|(x,y)|,$$
$$g(x,y) \ge M|(x,y)|, \tag{4.1}$$

for any $(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : 0 < |(x,y)| \le r$, where the constant M satisfies

$$\frac{M}{4} \inf_{1/4 \le t \le 3/4} \min\left\{ \int_{1/4}^{3/4} G(t,s) A(s)\, ds, \int_{1/4}^{3/4} G(t,s) B(s)\, ds \right\} > 1.$$

If $(\varphi, \psi) \in K_1 \times K_1$, $\|(\varphi, \psi)\| = r$, and $T(\varphi, \psi) = \lambda(\varphi, \psi)$ for some $\lambda > 0$, then we get

$$\lambda(\varphi(t), \psi(t)) = \left(\int_0^1 G(t,s) A(s) f(\varphi(s), \psi(s))\, ds, \right.$$
$$\left. \int_0^1 G(t,s) B(s) g(\varphi(s), \psi(s))\, ds \right).$$

Picking $t_0 \in [\frac{1}{4}, \frac{3}{4}]$ such that

$$|(\varphi(t_0), \psi(t_0)| = \inf_{t \in [1/4, 3/4]} |(\varphi(t), \psi(t))| > 0,$$

then

$$
\begin{aligned}
&\lambda|(\varphi(t_0), \psi(t_0))| \\
&\quad = \inf_{t\in[1/4,3/4]} \left| \left(\int_0^1 G(t,s)A(s)f(\varphi(s),\psi(s))\, ds, \right.\right. \\
&\qquad\qquad \left.\left. \int_0^1 G(t,s)B(s)g(\varphi(s),\psi(s))\, ds \right) \right| \\
&\quad \geq M \inf_{t\in[1/4,3/4]} \left| \left(\int_0^1 G(t,s)A(s)|(\varphi(s),\psi(s))|\, ds, \right.\right. \\
&\qquad\qquad \left.\left. \int_0^1 G(t,s)B(s)|(\varphi(s),\psi(x))|\, ds \right) \right| \\
&\quad \geq M \inf_{t\in[1/4,3/4]} \left| \left(\int_{1/4}^{3/4} G(t,s)A(s) \left| \left(\frac{\|\varphi\|}{4}, \frac{\|\psi\|}{4} \right) \right| ds, \right.\right. \\
&\qquad\qquad \left.\left. \int_{1/4}^{3/4} G(t,s)B(s) \left| \left(\frac{\|\varphi\|}{4}, \frac{\|\psi\|}{4} \right) \right| ds \right) \right| \\
&\quad = \frac{Mr}{4} \inf_{t\in[1/4,3/4]} \left| \left(\int_{1/4}^{3/4} G(t,s)A(s)\, ds, \int_{1/4}^{3/4} G(t,s)B(s)\, ds \right) \right| \\
&\quad > r
\end{aligned}
$$

and, hence, $\lambda > 1$.

If (H_4)(i) holds, say, $f(x,y) \leq N$ and $g(x,y) \leq N$, where N is a positive constant, then we choose

$$R > \max\left\{ 2r, N\int_0^1 G(t,s)A(s)\, ds, N\int_0^1 G(t,s)B(s)\, ds \right\}.$$

If $(\varphi,\psi) \in K_1 \times K_1$, $\|(\varphi,\psi)\| = R$, and

$$T(\varphi,\psi) = \lambda(\varphi,\psi) \qquad \text{for some } \lambda > 0,$$

then there exists $t_0 \in [0,1]$ such that

$$\|(\varphi,\psi)\| = |(\varphi(t_0), \psi(t_0))|$$

and, therefore,

$$
\begin{aligned}
\lambda\|(\varphi(t_0),\psi(t_0))\| &= \left| \left(\int_0^1 G(t_0,s)A(s)f(\varphi(s),\psi(s))\, ds, \right.\right. \\
&\qquad \left.\left. \int_0^1 G(t_0,s)B(s)g(\varphi(s),\psi(s))\, ds \right) \right| \\
&< R;
\end{aligned}
$$

i.e., $\lambda < 1$.

If (H_4)(ii) holds, then there is a $N > 0$ such that $g(x, y) \le N$. From (2.3), there is a $R \in \Sigma_f$, $R > \max\{2r, N\int_0^1 G(s,s)B(s)\,ds\}$ such that

$$f(x, y) \le \varepsilon|(x, y)| \qquad \text{for } (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+,\ |(x, y)| \ge R,$$

where the constant $\varepsilon > 0$ is such that

$$\varepsilon \int_0^1 G(s, s) A(s)\, ds < 1.$$

In particular, $|(x, y)| = R$ implies $f(x, y) \le \varepsilon R$.

Let $(\varphi, \psi) \in K_1 \times K_1$, $\|(\varphi, \psi)\| = R$, and

$$T(\varphi, \psi) = \lambda(\varphi, \psi) \qquad \text{for some } \lambda > 0.$$

Then there exists a $t_0 \in [0, 1]$ such that

$$\|(\varphi, \psi)\| = |(\varphi(t_0), \psi(t_0)|$$

and, therefore,

$$\begin{aligned}
&\lambda|(\varphi(t_0), \psi(t_0))| \\
&\quad = \left| \left(\int_0^1 G(t_0, s) A(s) f(\varphi(s), \psi(s))\, ds, \right. \right. \\
&\qquad\quad \left. \left. \int_0^1 G(t_0, s) B(s) g(\varphi(s), \psi(s))\, ds \right) \right| \\
&\quad \le \left| \left(\int_0^1 G(t_0, s) A(s) \max_{[0, R; 0, R]} f(x, y)\, ds, N \int_0^1 G(t_0, s) B(s)\, ds \right) \right| \\
&\quad < R
\end{aligned}$$

(we are able to do this since $R \in \Sigma_f$); i.e., $\lambda < 1$.

If (H_4)(iii) holds, then there is $R \in \Sigma_f$, $R > 2r$ such that

$$\begin{aligned}
f(x, y) &\le \varepsilon|(x, y)|, \\
g(x, y) &\le \varepsilon|(x, y)| \qquad \text{for } (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+,\ |(x, y)| \ge R,
\end{aligned}$$

where the constant ε satisfies

$$\varepsilon \max\left\{\int_0^1 G(s,s)A(s)\,ds, \int_0^1 G(s,s)B(s)\,ds\right\} < 1.$$

The monotonicity assumption on g implies that

$$R \in \Sigma_g$$

(cf. (3.5)).

Let $(\varphi, \psi) \in K_1 \times K_1$, $\|(\varphi, \psi)\| = R$, and

$$T(\varphi, \psi) = \lambda(\varphi, \psi) \qquad \text{for some } \lambda > 0.$$

Then there is $t_0 \in [0, 1]$ such that

$$\|(\varphi, \psi)\| = |(\varphi(t_0), \psi(t_0)|$$

and, therefore,

$$\begin{aligned}
\lambda|(\varphi(t_0), \psi(t_0))| &\leq \left|\left(\int_0^1 G(t_0, s)A(s) \max_{[0,R;0,R]} f(x,y)\,ds,\right.\right.\\
&\qquad \left.\left.\int_0^1 G(t_0, s)B(s) \max_{[0,R;0,R]} g(x,y)\,ds\right)\right|\\
&\leq \varepsilon R \max\left\{\int_0^1 G(s,s)A(s)\,ds, \int_0^1 G(s,s)B(s)\,ds\right\}\\
&< R;
\end{aligned}$$

i.e., $\lambda < 1$.

If (H_4)(iv) or (H_4)(v) holds, a simple modification of the above argument yields that there exists $R > 2r$, such that

$$(\varphi, \psi) \in K_1 \times K_1, \qquad \|(\varphi, \psi)\| = R, \qquad T(\varphi, \psi) = \lambda(\varphi, \psi) \Rightarrow \lambda < 1.$$

The last step is to prove that

$$\inf_{\|(\varphi, \psi)\| = r} \|T(\varphi, \psi)\| > 0.$$

From (4.1), we get that for $\|(\varphi, \psi)\| = r$,

$$\begin{aligned}
&\|T(\varphi, \psi)\| \\
&\quad \geq \left\| \left(\int_0^1 G(t,s) A(s) M |(\varphi(s), \psi(s))| ds, \right.\right. \\
&\qquad\qquad \left.\left. \int_0^1 G(t,s) B(s) M |(\varphi(s), \psi(s))|\, ds \right) \right\| \\
&\quad \geq M \left\| \left(\int_{1/4}^{3/4} G(t,s) A(s) \left| \left(\frac{\|\varphi\|}{4}, \frac{\|\psi\|}{4} \right) \right| ds, \right.\right. \\
&\qquad\qquad \left.\left. \int_{1/4}^{3/4} G(t,s) B(s) \left| \left(\frac{\|\varphi\|}{4}, \frac{\|\psi\|}{4} \right) \right| ds \right) \right\| \\
&\quad \geq \frac{Mr}{4} \min\left\{ \int_{1/4}^{3/4} G(t,s) A(s)\, ds, \int_{1/4}^{3/4} G(t,s) B(s)\, ds \right\},
\end{aligned}$$

and the last expression is a positive number independent of (φ, ψ).

Thus T has a fixed point $(\overline{\varphi}, \overline{\psi})$ in

$$D = \left\{ (\varphi, \psi) \in K_1 \times K_1 | r \leq \|(\varphi, \psi)\| \leq R \right\}.$$

We claim that $\overline{\varphi}(t) > 0$, $\overline{\psi}(t) > 0$ for $t \in (0,1)$. Indeed, from $\|(\overline{\varphi}, \overline{\psi})\| \geq r$ and Lemma 4, we know that $\overline{\varphi} > 0$ in $(0,1)$ or $\overline{\psi} > 0$ in $(0,1)$. We may assume that $\overline{\psi} > 0$ in $(0,1)$. Again from $\|(\overline{\varphi}, \overline{\psi})\| > r$, there is $s_0 \in (0,1)$ such that $|(\overline{\varphi}(s_0), \overline{\psi}(s_0))| = r$ and $|(\overline{\varphi}(s), \overline{\psi}(s))| \leq r$ for $0 < s \leq s_0$. This, together with (4.1) and Lemma 4, implies

$$\begin{aligned}
&f\big(\overline{\varphi}(s), \overline{\psi}(s)\big) \geq M \big|\big(\overline{\varphi}(s), \overline{\psi}(s)\big| > 0, \\
&g\big(\overline{\varphi}(s), \overline{\psi}(s)\big) \geq M \big|\big(\overline{\varphi}(s), \overline{\psi}(s)\big)\big| > 0, \qquad 0 < s < s_0,
\end{aligned}$$

and, hence,

$$\begin{aligned}
\overline{\varphi}(t) &= \int_0^1 G(t,s) A(s) f\big(\overline{\varphi}(s), \overline{\psi}(s)\big)\, ds \\
&\geq \int_0^{s_0} G(t,s) A(s) f\big(\overline{\varphi}(s), \overline{\psi}(s)\big)\, ds > 0, \qquad 0 < t < 1.
\end{aligned}$$

Therefore (1.1)–$(1.2)_a$ has a positive radial solution. This completes the proof of the first part of Theorem 1.

Next we consider (3.3)–(3.4)$_b$ and (3.3)–(3.4)$_c$. It is easy to check that (3.3)–(3.4)$_b$ and (3.3)–(3.4)$_c$ are equivalent to the integral equations

$$(\varphi(t), \psi(t)) = \left(\int_0^1 K_1(t,s) A(s) f(\varphi(s), \psi(s))\, ds, \right.$$
$$\left. \int_0^1 K_1(t,s) B(s) g(\varphi(s), \psi(s))\, ds \right)$$

and

$$(\varphi(t), \psi(t)) = \left(\int_0^1 K_2(t,s) A(s) f(\varphi(s), \psi(s))\, ds, \right.$$
$$\left. \int_0^1 K_2(t,s) B(s) g(\varphi(s), \psi(s))\, ds \right),$$

respectively, where

$$K_1(t,s) = \begin{cases} t, & t \le s, \\ s, & t > s, \end{cases} \qquad K_2(t,s) = \begin{cases} 1-s, & t \le s, \\ 1-t, & t > s. \end{cases}$$

For (3.3)–(3.4)$_b$ let P_1 be the cone

$$P_1 = \left\{ z \in X | z(t) \ge 0,\ z(0) = z'(1) = 0,\ \min_{1/2 \le t \le 1} z(t) \ge \frac{\|z\|}{2} \right\};$$

for (3.3)–(3.4)$_c$ let P_2 be the cone

$$P_2 = \left\{ z \in X | z(t) \ge 0,\ z'(0) = z(1) = 0,\ \min_{0 \le t \le 1/2} z(t) \ge \frac{\|z\|}{2} \right\}.$$

By the method above, the fixed-point theorem is used to prove that both (1.1)–(1.2)$_b$ and (1.1)–(1.2)$_c$ have positive solutions. This completes the proof of Theorem 1. ∎

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